Université Ibn Khaldoun Tiaret.
Département des Mathématiques
Faculté des Mathématiques et Informatique

$2^{\text{ère}}$ anné Master L.M.D.(Maths AFED)
Année universitaire : 2025-2026
Le 24-01-2026

Correction of the exam EDO3,

Consider the boundary value problem for fractional differential equations,

$$^cD^\alpha y(t) = f(t,y), \text{ for each } t \in J = [0,T], \quad 2 < \alpha \leq 3, \tag{1}$$

$$y(0) = y_0, \quad y'(0) = y_0^*, \quad y''(T) = y_T \tag{2}$$

***Question 1* :** (5points)
Assume y satisfies (1), then a Lemma implies that

$$y(t) = c_0 + c_1 t + c_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds.$$

From (2), a simple calculation gives

$$c_0 = y_0, \ c_1 = y_0^*,$$

and

$$y''(T) = 2c_2 + \frac{1}{\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} h(s) ds = y_T.$$

Hence we get direct equation . Inversely, it is clear that if y satisfies direct equation, then equations (1)-(2) hold.

***Question 2* :** (5points)
Transform the problem (1)–(2) into a fixed point problem. Consider the operator

$$N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$$

defined by

$$\begin{aligned} N(y)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s)) ds \\ &- \frac{t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} f(s,y(s)) ds \\ &+ y_0 + y_0^* t + \frac{y_T}{2} t^2. \end{aligned}$$

Clearly, the fixed points of the operator N are solution of the problem . We shall use the Banach contraction principle to prove that N has a fixed point. We shall show that N *is a contraction.*

Let $x, y \in C(J,)$. Then, for each $t \in J$ we have

$$\begin{aligned} |N(x)(t) - N(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| ds \\ &+ \frac{T^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} |f(s,x(s)) - f(s,y(s))| ds \\ &\leq \frac{k\|x-y\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &+ \frac{T^2 k\|x-y\|_\infty}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-1} ds \\ &\leq kT^\alpha \left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} \right] \|x-y\|_\infty. \end{aligned}$$

Thus

$$\|N(x) - N(y)\|_\infty \leq kT^\alpha \left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)} \right] \|x-y\|_\infty.$$

Consequently N is a contraction. As a consequence of Banach fixed point theorem, we deduce that N has a fixed point which is a solution of the problem $(1) - (2)$. □

Question 3 : (10points)

Proof. We shall use Schaefer's fixed point theorem to prove that N has a fixed point. It can be easily shown that N is continuous and completely continuous.

Now it remains to show that the set

$$\mathcal{E} = \{y \in C(J, \mathbb{R}) : y = \lambda N(y) \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}$, then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$\begin{aligned} y(t) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s)) ds \\ &- \frac{\lambda t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} f(s,y(s)) ds \\ &+ \lambda y_0 + \lambda y_0^* t + \lambda \frac{y_T}{2} t^2. \end{aligned}$$

This implies by (H3) that for each $t \in J$ we have

$$\begin{aligned} |y(t)| &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &+ \frac{MT^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} ds + |y_0| + |y_0^*| T + \frac{|y_T|}{2} T^2. \\ &\leq \frac{M}{\alpha\Gamma(\alpha)} T^\alpha + \frac{M}{(\alpha-2)\Gamma(\alpha-2)} T^\alpha + |y_0| + |y_0^*| T + \frac{|y_T|}{2} T^2. \end{aligned}$$

Thus for every $t \in J$, we have

$$\|y\|_\infty \leq \frac{M}{\Gamma(\alpha+1)}T^\alpha + \frac{M}{\Gamma(\alpha-1)}T^\alpha + |y_0| + |y_0^*|T + \frac{|y_T|}{2}T^2 := R.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem $(1)-(2)$. □