Ibn Khaldoun University – Tiaret
Faculty of Mathematics and Computer Science.

Department of Computer Science
Module: Algebra 01 (1^{st} LMD Year 2025-2026)
Duration: 2 hour.

Final Exam

EXERCISE 01 (06 pts):

1) Let $a, b \in \mathbb{Z}$. Show by contrapositive that :
   (4 does not divide ab) $\Longrightarrow$ (a *is odd* *or* b is odd)

2) Let $x, y \in \left]-\infty, 0\right]$, show by contradiction that $\left(\frac{x}{1-y} \neq \frac{y}{1-x}\right) \vee (x = y)$.

3) Let $n \in \mathbb{Z}$. Show by cases that $(n-1)(n+2)$ is even.

EXERCISE 02 (08 pts):

1) Let $f : E \longrightarrow F$, $g : F \longrightarrow G$ be two applications. Show that :
   ($g \circ f$ is surjective and g is injective) $\Longrightarrow$ f is surjective

2) Let $f : [-3, 3] \longrightarrow \mathbb{R}$ be the application defined by : $f(x) = \sqrt{3 - |x|}$
   2.1. Determine $f(\{-1,0,1\})$ and $f^{-1}(\{0,2\})$
   2.2. f Is it injective ? is it surjective ? is it bijective ? Justify your answer.

3) Let $g : [-3, 0] \longrightarrow \mathbb{R}$ be an application defined by : $g(x) = f(x)$.
   3.1. Show that g is injective and not surjective.
   3.2. Deduce that $g \circ g$ is not surjective.

4) Show that the application $h : [-3, 0] \longrightarrow [0, \sqrt{3}]$, $h(x) = g(x)$ is bijective , then determine its inverse h^{-1}.

EXERCISE 03 (06 pts):

Let $\mathcal{R}$ be the relation defined on $\mathbb{Z} \times \mathbb{Z}^*$ by :
$\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}^* : \quad (a, b)\mathcal{R}(c, d) \Leftrightarrow \frac{a}{b} - \frac{c}{d} = 0$.

1. Show that $\mathcal{R}$ is an equivalence relation .
2. Determine $\widehat{(1,1)}$, $\widehat{(0,2)}$ the equivalence classes of $(1,1)$ and $(0,2)$.

Good luck

Algebra 01 (1ère LMD Year 2024-2025)

Final Exam Answer Key

EXERCISE 01 (06 pts):

1) Let $a, b \in \mathbb{Z}$. To show by contrapositive that : (4 does not divide ab) $\Longrightarrow$ (a *is odd* *or* b is odd) ,
It suffices to show that : (a is even and b is even) $\Longrightarrow$ (4 divides ab) , we have :

(a is even and b is even) $\Longrightarrow (\exists k, k' \in \mathbb{Z} : a = 2k \ , \ b = 2k') \Longrightarrow (\exists k, k' \in \mathbb{Z} : ab = 4kk')$
$\Longrightarrow$ (4 divides ab)

Consequently (4 does not divide ab) $\Longrightarrow$ (a *is odd* *or* b is odd) …………………………… (2 pts)

2) Let $x, y \in]-\infty, 0]$. Assume that $\left(\frac{x}{1-y} = \frac{y}{1-x}\right) \wedge (x \neq y)$ is true , ………………………………(0.5pt)

so , $x - x^2 - y + y^2 = 0 \ \wedge \ x \neq y$ i.e. $(x-y)(x+y-1) = 0 \ \wedge \ x \neq y$

hence $x = y \ \wedge \ x \neq y$ because $x, y \in]-\infty, 0]$, which is absurd.

As a result $\forall x, y \in]-\infty, 0]$, $\left(\frac{x}{1-y} \neq \frac{y}{1-x}\right) \vee (x = y)$. ……………………………………………………(1.5pt)

Second method: $(x-y)(x+y-1) = 0 \ \wedge \ x \neq y$, thus $x + y = 1$, which is absurd , because $x, y \in]-\infty, 0]$.

3) Let $n \in \mathbb{Z}$. we have :

1st case: If $n = 2k$, with $k \in \mathbb{Z}$, so ,

$$(n-1)(n+2) = (2k-1)(2k+2) = 2(2k-1)(k+1) \text{ , which is even.}$$

2nd case: If $n = 2k+1$, with $k \in \mathbb{Z}$, so ,

$$(n-1)(n+2) = (2k+1-1)(2k+1+2) = 2k(2k+3) \text{ , which is even.}$$

Therefore, in all cases $(n-1)(n+2)$ is even ……………………………………………………(2pts)

Exercise 02 (08 pts):

1) Suppose that $g \circ f$ is surjective and g is injective and let's show that f is surjective.
Let $y \in F$, then $g(y) \in G$ and since $g \circ f$ is surjective , there exists $x \in E$ such that
$g(y) = (g \circ f)(x)$, i.e $g(y) = g\big(f(x)\big)$, and since g is injective , then $y = f(x)$.
Therefore, there exists $x \in E$ such that $y = f(x)$, hence f is surjective. …………………………………(1pt)

2.1.1) We have : $f(-1) = \sqrt{2}$, $f(0) = \sqrt{3}$ and $f(1) = \sqrt{2}$.

Then $f(\{-1, 0, 1\}) = \{\sqrt{2}, \sqrt{3}\}$.……………………………………………………………………(0.5pt)

2.1.2) $f(x) = 0 \Leftrightarrow \sqrt{3 - |x|} = 0 \Leftrightarrow 3 - |x| = 0 \Leftrightarrow |x| = 3 \Leftrightarrow (x = -3 \ \vee \ x = 3)$

$f(x) = 2 \Leftrightarrow \sqrt{3 - |x|} = 2 \Leftrightarrow 3 - |x| = 4 \Leftrightarrow |x| = -1$, which has no real solution.

So $f^{-1}(\{0,2\}) = \{-3, 3\}$. …………………………………………………………………………(0.5pt)

2.2) According to 2.1.1) , we have $f(-1) = \sqrt{2} = f(1)$ and $-1 \neq 1$ therefore f is not injective…….(0.5pt)

According to 2.1.2) , we have $y = 2$ has no preimage, then f is not surjective . ……………………(0.5pt)

3) For $x \in [-3, 0]$, we have $|x| = -x$ and $g(x) = f(x) = \sqrt{3 + x}$.

3.1) Let $x, x' \in [-3\,,0]$ we have :

$g(x) = g(x') \implies \sqrt{3+x} = \sqrt{3+x'} \implies 3+x = 3+x' \implies x = x'$.

Then g is injective . …………………………………………………………………………………………(1pt)

From 2.1.2), we have : $y = 2$ has no preimage by f in $[-3\,,3]$, so it has no preimage by g in $[-3\,,0]$ (The equation $f(x) = 2$ has no solutions in $[-3\,,3]$, therefore it has no solutions in $[-3\,,0]$).

Then g is not surjective. ……………………………………………………………………………………(0.5pt)

3.2) Suppose that $g \circ g$ is surjective and since g is injective , then, according to the 1st question , g is surjective , this is not the case.

Therefore $g \circ g$ is not surjective. ……………………………………………………………………………(1pt)

4.1) Since g is injective , then h is injective. ……………………………………………………………(0.5pt)

4.2) Let $y \in [0\,,\sqrt{3}]$, Let's find $x \in [-3\,,0]$, such that $y = h(x)$, we have :

$y = h(x) \Leftrightarrow y = \sqrt{3+x} \Leftrightarrow y^2 = 3 + x \Leftrightarrow x = y^2 - 3.$

For $0 \leq y \leq \sqrt{3}$, we have $0 \leq y^2 \leq 3$ so $-3 \leq y^2 - 3 \leq 0$.

Then it suffices to take $x = y^2 - 3 \in [-3\,,0]$, to have $y = h(x)$.

Therefore, h is surjective. ………………………………………………………………………………………(1pt)

We conclude that h is bijective.

4.3) Since h is bijective, then h^{-1} exists and is bijective, and we have :

$h^{-1} : [0\,,\sqrt{3}] \longrightarrow [-3\,,0]$ such that $h^{-1}(y) = x = y^2 - 3.$ ……………………………(1pt)

EXERCISE 03 (06 points):

1.1) Let $(a,b) \in \mathbb{Z} \times \mathbb{Z}^*$, we have $\frac{a}{b} - \frac{a}{b} = 0$ i.e. $(a,b)\,\mathcal{R}\,(a,b)$.

Then $\mathcal{R}$ it is reflexive. ………………………………………………………………………………………(1pt)

1.2) Let $(a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}^*$, we have :

$$\begin{aligned}(a,b)\,\mathcal{R}\,(c,d) &\implies \frac{a}{b} - \frac{c}{d} = 0\\ &\implies \frac{c}{d} - \frac{a}{b} = 0\\ &\implies (c,d)\mathcal{R}(a,b)\end{aligned}$$

Then $\mathcal{R}$ it is symmetric. ………………………………………………………………………………………(1pt)

1.3) Let $(a,b),(c,d),(m,n) \in \mathbb{Z} \times \mathbb{Z}^*$, we have :

$$\begin{aligned}\big((a,b)\,\mathcal{R}\,(c,d) \text{ and } (c,d)\,\mathcal{R}\,(m,n)\big) &\implies \left(\frac{a}{b} - \frac{c}{d} = 0 \text{ and } \frac{c}{d} - \frac{m}{n} = 0\right)\\ &\implies \left(\frac{a}{b} - \frac{c}{d}\right) + \left(\frac{c}{d} - \frac{m}{n}\right) = 0\\ &\implies \frac{a}{b} - \frac{m}{n} = 0\\ &\implies (a,b)\,\mathcal{R}\,(m,n)\end{aligned}$$

Then $\mathcal{R}$ it is transitive. ………………………………………………………………………………………(1.5pt)

We conclude that $\mathcal{R}$ is an equivalence relation. ……………………………………………………(0.5pt)

2) $\dot{\widehat{(1,1)}} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ \ (a,b)\ \mathcal{R}\ (1,1)\ \} = \left\{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ \ \frac{a}{b} - \frac{1}{1} = \ 0\right\}$

$= \left\{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ \ \frac{a}{b} = 1\right\} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ \ \ a = b\ \wedge\ a \neq 0\ \}$

$= \{(b,b)\ \ /\ \ b \in \mathbb{Z}^*\}$ ... . (1pt)

$\dot{\widehat{(0,2)}} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ (a,b)\ \mathcal{R}\ (0,2)\ \} = \left\{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*,\ \frac{a}{b} - \frac{0}{2} = \ 0\right\}$

$= \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ \ a = 0 \times b\ \} = \{(0,b)\ \ /\ \ b \in \mathbb{Z}^*\}$........................ ....(1pt)

Université Ibn Khaldoun – Tiaret
Faculté des Mathématiques et de l'Informatique

Département d'Informatique
Module : Algèbre 01 (1$^{\text{ère}}$ Année LMD 2025-2026)
Durée : 2^{h}

Examen Final

EXERCICE 01 (06 pts) :

1) Soit $a, b \in \mathbb{Z}$. Montrer par contraposée que :
(4 ne divise pas ab) $\Longrightarrow$ (a est impair ou b est impair)
2) Soient $x, y \in]-\infty, 0]$, montrer par l'absurde que $\left(\frac{x}{1-y} \neq \frac{y}{1-x}\right) \vee (x = y)$.
3) Soit $n \in \mathbb{Z}$. Montrer par disjonction des cas que $(n-1)(n+2)$ est pair.

EXERCICE 02 (08pts) :

1) Soit $f: E \longrightarrow F$, $g: F \longrightarrow G$ deux applications. Montrer que :
($g \circ f$ est surjective et g est injective) $\Longrightarrow$ f est surjective.
2) Soit $f : [-3, 3] \longrightarrow \mathbb{R}$ l'application définie par : $f(x) = \sqrt{3 - |x|}$
   2.1. Déterminer $f(\{-1,0,1\})$ *et* $f^{-1}(\{0,2\})$
   2.2. f est-elle injective ? Est-elle surjective ? Est-elle bijective ? Justifier
3) Soit $g : [-3, 0] \longrightarrow \mathbb{R}$ l'application définie par : $g(x) = f(x)$.
   3.1. Montrer que g est injective et n'est pas surjective.
   3.2. En déduire que $g \circ g$ n'est pas surjective.
4) Montrer que l'application $h : [-3, 0] \longrightarrow [0, \sqrt{3}]$, $h(x) = g(x)$ est bijective, puis déterminer sa réciproque h^{-1}.

EXERCICE 03 (06pts) :

Soit $\mathcal{R}$ la relation définie sur $\mathbb{Z} \times \mathbb{Z}^*$ par :
$\forall (a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}^* : \quad (a,b)\mathcal{R}(c,d) \Leftrightarrow \frac{a}{b} - \frac{c}{d} = 0$.

1. Montrer que $\mathcal{R}$ est une relation d'équivalence.
2. Déterminer $\dot{\widehat{(1,1)}}$, $\dot{\widehat{(0,2)}}$ les classes d'équivalence de $(1,1)$ *et* $(0,2)$.

Bon courage

Algèbre 01 (1ère Année LMD 2024-2025)

Corrigé de l'Examen final

EXERCICE 01 (06 pts) :

1) Soit $a, b \in \mathbb{Z}$. Pour montrer par contraposée que : (4 ne divise pas ab) $\Longrightarrow$ (a est impair ou b est impair) , Il suffit de montrer que : (a est pair et b est pair) $\Longrightarrow$ (4 divise ab) , On a :

(a est pair et b est pair) $\Longrightarrow (\exists k, k' \in \mathbb{Z} :\ a = 2k\ ,\ b = 2k') \Longrightarrow (\exists k, k' \in \mathbb{Z} :\ ab = 4kk')$
$\Longrightarrow$ (4 divise ab)

Par suite (4 ne divise pas ab) $\Longrightarrow$ (a est impair ou b est impair) ……………………………………(2pts)

2) Soit $x, y \in \left]-\infty, 0\right]$. Supposons que $\left(\frac{x}{1-y} = \frac{y}{1-x}\right) \wedge (x \neq y)$ est vraie , ……………………………(0.5pt)

donc : $x - x^2 - y + y^2 = 0\ \wedge\ x \neq y$ c-à-d $(x-y)(x+y-1) = 0\ \wedge\ x \neq y$

d'où $x = y\ \wedge\ x \neq y$ car $x, y \in \left]-\infty, 0\right]$, ce qui est absurde .

Par suite $\forall x, y \in \left]-\infty, 0\right]$, $\left(\frac{x}{1-y} \neq \frac{y}{1-x}\right) \vee (x = y)$. ……………………………………………………(1.5pt)

Deuxième méthode : $(x-y)(x+y-1) = 0\ \wedge\ x \neq y$, ainsi $x + y = 1$, ce qui est absurde, car $x, y \in \left]-\infty, 0\right]$.

3) Soit $n \in \mathbb{Z}$. On a :

1er cas : Si $n = 2k,\ avec\ k \in \mathbb{Z}$, alors ,

$(n-1)(n+2)\ = (2k-1)(2k+2)\ = 2(2k-1)(k+1)$, qui est pair.

2ème cas : Si $n = 2k+1,\ avec\ k \in \mathbb{Z}$, alors :

$(n-1)(n+2)\ = (2k+1-1)(2k+1+2)\ = 2k(2k+3)$, qui est pair.

Par suite, dans tous les cas $(n-1)(n+2)$ est pair.…………………………………………………(2pts)

Exercice 02 (08pts):

1) Supposons que $g \circ f$ est surjective et g est injective et montrons que f est surjective.

Soit $y \in F$, alors $g(y) \in G$ et puisque $g \circ f$ est surjective , il existe $x \in E$ tel que $g(y) = (g \circ f)(x)$, c-à-d $g(y) = g\big(f(x)\big)$, et puisque g est injective , alors $y = f(x)$.

Donc il existe $x \in E$ tel que $y = f(x)$, d'où f est surjective. ……………………………………(1pt)

2.1.1) On a : $f(-1) = \sqrt{2}$, $f(0) = \sqrt{3}$ et $f(1) = \sqrt{2}$.

Alors $f(\{-1, 0, 1\}) = \{\sqrt{2}, \sqrt{3}\}$. ……………………………………………………………(0.5pt)

2.1.2) $f(x) = 0 \Leftrightarrow \sqrt{3-|x|} = 0 \Leftrightarrow 3 - |x| = 0 \Leftrightarrow |x| = 3 \Leftrightarrow (x = -3\ \vee\ x = 3)$,

$f(x) = 2 \Leftrightarrow \sqrt{3-|x|} = 2 \Leftrightarrow 3 - |x| = 4 \Leftrightarrow |x| = -1$, qui n'a pas de solutions réelles.

Alors $f^{-1}(\{0,2\}) = \{-3\ ,\ 3\}$. ……………………………………………………………(0.5pt)

2.2) D'après 2.1.1), on a : $f(-1) = \sqrt{2} = f(1)$ et $-1 \neq 1$ donc f n'est pas injective.…………(0.5pt)

D'après 2.1.2) , on a : $y = 2$ n'a pas d'antécédent, alors f n'est pas surjective. ……………………(0.5pt)

3) Pour $x \in [-3, 0]$, on a $|x| = -x$ et $g(x) = f(x) = \sqrt{3+x}$

3.1) Soient $x, x' \in [-3, 0]$, on a :

$g(x) = g(x') \implies \sqrt{3+x} = \sqrt{3+x'} \implies 3 + x = 3 + x' \implies x = x'$.

Alors g est injective. ……………………………………………………………………………………(1pt)

D'après 2.1.2) , on a $y = 2$ n'a pas d'antécédent par f dans $[-3, 3]$, donc il n'a pas d'antécédent par g dans $[-3, 0]$ (l'équation $f(x) = 2$ n'a pas de solutions dans $[-3, 3]$, donc elle n'a pas de solutions dans $[-3, 0]$).

alors g n'est pas surjective. ……………………………………………………………………………(0.5pt)

3.2) Supposons que $g \circ g$ est surjective et puisque g est injective , alors d'après la 1ere question , g est Surjective , ce qui n'est pas le cas.

Alors $g \circ g$ n'est pas surjective. ……………………………………………………………………(1pt)

4.1) Puisque g est injective , alors h est injective. ………………………………………………………(0.5pt)

4.2) Soit $y \in [0, \sqrt{3}]$, cherchons $x \in [-3, 0]$, tel que $y = h(x)$, on a :

$y = h(x) \Leftrightarrow y = \sqrt{3+x} \Leftrightarrow y^2 = 3 + x \Leftrightarrow x = y^2 - 3.$

Pour $0 \leq y \leq \sqrt{3}$, on a $0 \leq y^2 \leq 3$ donc $-3 \leq y^2 - 3 \leq 0$.

Alors il suffit de prendre $x = y^2 - 3 \in [-3, 0]$, pour avoir $y = h(x)$.

Par suite h est surjective. ……………………………………………………………………………(1pt)

On conclut que h est bijective.

4.3) Puisque h est bijective alors h^{-1} existe et bijective et on a :

$h^{-1} : [0, \sqrt{3}] \longrightarrow [-3, 0]$ telle que $h^{-1}(y) = x = y^2 - 3$ ……………………………………(1pt)

EXERCICE 03 (06pts):

1.1) Soit $(a, b) \in \mathbb{Z} \times \mathbb{Z}^*$, on a $\frac{a}{b} - \frac{a}{b} = 0$, c-à-d : $(a, b)\, \mathcal{R}\, (a, b)$.

Alors $\mathcal{R}$ est réflexive. ……………………………………………………………………………(1pt)

1.2) Soient $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}^*$, on a :

$$(a, b)\, \mathcal{R}\, (c, d) \implies \frac{a}{b} - \frac{c}{d} = 0$$
$$\implies \frac{c}{d} - \frac{a}{b} = 0$$
$$\implies (c, d) \mathcal{R} (a, b)$$

Alors $\mathcal{R}$ est symétrique.…………………………………………………………………………(1pt)

1.3) Soient $(a, b), (c, d), (m, n) \in \mathbb{Z} \times \mathbb{Z}^*$, on a :

$$\big((a, b)\, \mathcal{R}\, (c, d) \text{ et } (c, d)\, \mathcal{R}\, (m, n)\big) \implies \left(\frac{a}{b} - \frac{c}{d} = 0 \quad et \quad \frac{c}{d} - \frac{m}{n} = 0\right)$$
$$\implies \frac{a}{b} - \frac{c}{d} + \frac{c}{d} - \frac{m}{n} = 0$$
$$\implies \frac{a}{b} - \frac{m}{n} = 0$$
$$\implies (a, b)\, \mathcal{R}\, (m, n).$$

Alors $\mathcal{R}$ est transitive. ……………………………………………………………………………(1.5pt)

On conclut que $\mathcal{R}$ est une relation d'équivalence. ………………………………………………(0.5pt)

2) $\dot{\widehat{(1,1)}} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ (a,b)\ \mathcal{R}\ (1,1)\ \} = \left\{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\quad \frac{a}{b} - \frac{1}{1} = 0\right\}$

$= \left\{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\quad \frac{a}{b} = 1\right\} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\quad a = b\ \wedge\ a \neq 0\ \}$

$= \{(b,b)\ /\ b \in \mathbb{Z}^*\}$ ... . (1pt)

$\dot{\widehat{(0,2)}} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ (a,b)\ \mathcal{R}\ (0,2)\ \} = \left\{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*,\ \frac{a}{b} - \frac{0}{2} = 0\right\}$

$= \{(a,b) \in \mathbb{Z} \times \mathbb{Z}^*\ ,\ a = 0 \times b\ \} = \{(0,b)\ /\ b \in \mathbb{Z}^*\}$........................ ....(1pt)